Carman S. Cater

Problem 1. *A sphere of radius* 1 *meter centered at the origin is rotating about the* $z-$*axis with an angular speed of one radian per minute so that the point of the sphere located at* $(1, 0, 0)$ *at time* $t = 0$ *is located at* $(0, 1, 0)$ *at time* $t = \dfrac{\pi}{2}$ *minutes. A ladybug starts walking at time* $t = 0$ *from the north pole towards the south pole along a meridian with a constant speed such that at time* $t = \dfrac{\pi}{2}$ *minutes she will be crossing the equator. Find parametric equations for the ladybug's trajectory in* $\mathbb{R}^3$.

Write an integral that represents the total length of the ladybug's trajectory from the north pole to the south pole.

Solution. A nice parametric representation of the unit sphere that works well for us here is given by

$$X(\varphi, \theta) = (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)$$

with $0 < \varphi < \pi$, $0 < \theta < 2\pi$. Note in this parametrization φ is the angle with the positive z-axis, while θ is the angle with the positive x-axis.

Setting $\varphi(t) = t$ and $\theta(t) = t$ we see that $\frac{d\varphi}{dt} = 1$, $\frac{d\theta}{dt} = 1$. Therefore the rotation around the z-axis has constant angular speed of one radian per minute and our ladybug will travel the meridian with constant speed. Setting $\varphi(t) = \frac{\pi}{2}$ we see that $X(\frac{\pi}{2}, 0) = (1, 0, 0)$ and $X(\frac{\pi}{2}, \frac{\pi}{2}) = (0, 1, 0)$. This shows us the point of the sphere located at $(1, 0, 0)$ at time $t = 0$ is located at $(0, 1, 0)$ at time $t = \dfrac{\pi}{2}$.

For the parametrization of the ladybug starting from the north pole and walking towards the south pole along a meridian with constant speed we take

$$\alpha(t) = X(t, t) = (\sin t\cos t, \sin^2 t, \cos t)$$

with $0 \le t \le \pi$. At time $t = 0$ the ladybug is at the north pole $\alpha(0) = (0, 0, 1)$, at time $t = \frac{\pi}{2}$ it is crossing the equator $\alpha(\frac{\pi}{2}) = (0, 1, 0)$ and at time $t = \pi$ it has arrived at the south pole $\alpha(\pi) = (0, 0, -1)$.

We compute the length of the ladybug's trajectory from the north pole to the south pole in two different ways.

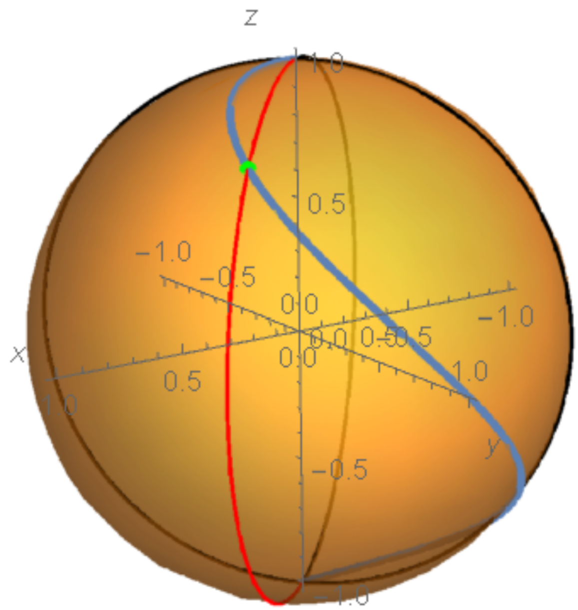


Figure 1: The ladybug (green) walking along the meridian (red) traces the curve (blue) as the sphere rotates counterclockwise about the z-axis with angular speed 1 radian per minute.

The first approach is to compute the first fundamental form of the surface

$$
\begin{aligned}
X &= (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi) \\
X_\varphi &= (\cos\varphi\cos\theta, \cos\varphi\sin\theta, -\sin\varphi) \\
X_\theta &= (-\sin\varphi\sin\theta, \sin\varphi\cos\theta, 0) \\
E &= X_\varphi \cdot X_\varphi = 1 \\
F &= X_\varphi \cdot X_\theta = 0 \\
G &= X_\theta \cdot X_\theta = \sin^2\varphi
\end{aligned}
$$

and take $\varphi(t) = t$ and $\theta(t) = t$. Thus our desired integral is given by

$$
\begin{aligned}
\mathcal{L} &= \int_I \sqrt{E\left(\frac{d\varphi}{dt}\right)^2 + 2F\frac{d\varphi}{dt}\frac{d\theta}{dt} + G\left(\frac{d\theta}{dt}\right)^2}\, dt \\
&= \int_0^\pi \sqrt{1+\sin^2 t}\, dt \approx 3.8202
\end{aligned}
$$

The second approach is to integrate the speed of $\alpha(t) = X(t,t) = (\sin t \cos t, \sin^2 t, \cos t)$ for $0 \leq t \leq \pi$. Using Mathematica to assist in the computation we see

$$\begin{aligned}
\mathcal{L} &= \int_0^\pi |\alpha'(t)| \; dt \\
&= \frac{1}{\sqrt{2}} \int_0^\pi \sqrt{3 - \cos 2t} \; dt \approx 3.8202
\end{aligned}$$

where the units are in meters. Note that of course these two definite integrals give the same value, the integrands are equivalent by the double angle identity for cosine.

Problem 2. *A surface S is generated by a horizontal (parallel to the $xy-$plane) line that rotates counterclockwise around the $z-$axis with an angular speed of one radian per minute at the same time that the horizontal plane that contains it is moving upward (remaining parallel to the $xy-$plane with a speed of one meter per minute. Find parametric equations for the surface S assuming that at time $t = 0$ the line coincides with the $x-$axis.*

Solution. This is a description of a helicoid. The surface S is generated by the parametrization

$$X(s, t) = (0, 0, t) + s(\cos t, \sin t, 0)$$

with $s, t \in \mathbb{R}$. It is clear that as the parameter t varies our point rotates counterclockwise around the z-axis with an angular speed of one radian per minute, while at the same time moving upward in the direction of the z-axis with a speed of one meter per minute.

At the time $t = 0$ the line coincides with the x-axis as seen by

$$X(s, 0) = (s, 0, 0)$$

so that as the parameter s varies we get a line coinciding with the x-axis.

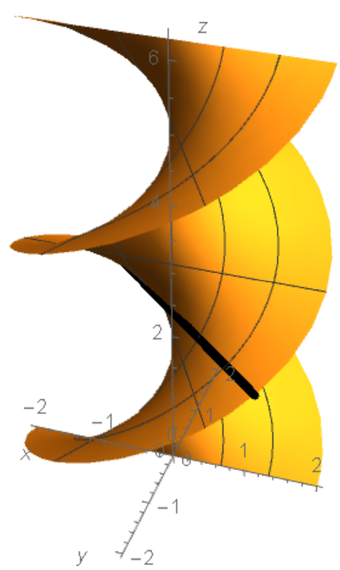


Figure 2: A picture of the helicoid being generated by the black line as it moves upward while rotating counterclockwise around the z-axis with $s \in [-2, 2]$ and $t \in [0, 2\pi]$.

Problem 3. *A curve* γ *is given by* $\gamma(t) = (\cos t, t, \cos t)$ *with* $0 \leq t \leq 2\pi$.

(a) *Decide wheter or not the parameter* t *is arc length parameter for* γ.

(b) *Prove that the length of* γ *is* $\mathcal{L}(\gamma) = \int_0^{2\pi} \sqrt{1 + 2\sin^2 t}\, dt$

(c) *Find the torsion of the curve at the point* $\gamma(\pi)$.

Solution (a). No, the parameter t is not an arc length parameter for γ. We compute $\gamma'(t) = (-\sin t, 1, -\sin t)$ and check the speed of γ and notice that

$$|\gamma'(t)| = \sqrt{1 + 2\sin^2 t}$$

which is definitely not a constant of 1.

Solution (b). The length of the curve γ is given by

$$\begin{aligned}\mathcal{L}(\gamma) &= \int_a^b |\gamma'(t)|\ dt \\ &= \int_0^{2\pi} \sqrt{1 + 2\sin^2 t}\ dt \approx 8.73775\end{aligned}$$

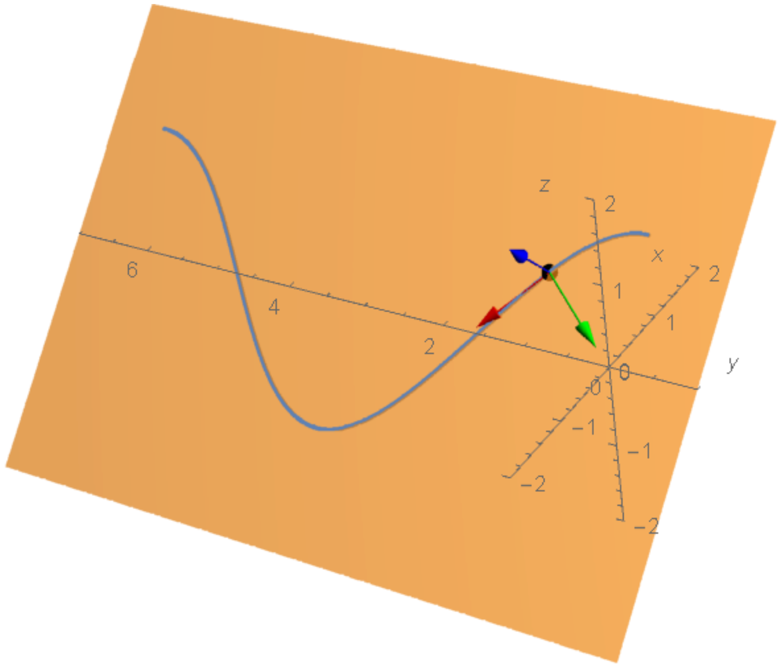
Solution (c). We will use the formula provided in the appendix

$$\tau(t) = \frac{(\gamma' \times \gamma'') \cdot \gamma'''}{|\gamma' \times \gamma''|^2}$$

The necessary computations are

$$\begin{aligned}\gamma &= (\cos t, t, \cos t) \\ \gamma' &= (-\sin t, 1, -\sin t) \\ \gamma'' &= (-\cos t, 0, -\cos t) \\ \gamma''' &= (\sin t, 0, \sin t) \\ \gamma' \times \gamma'' &= (-\cos t, 0, \cos t) \\ (\gamma' \times \gamma'') \cdot \gamma''' &= 0 \\ |\gamma' \times \gamma''|^2 &= 2\cos^2 t\end{aligned}$$

Therefore the torsion is zero everywhere, yet at the points $(2n-1)\frac{\pi}{2}$ with $n \in \mathbb{Z}^+$ it is undefined. Thus $\tau(\pi) = 0$. This implies that γ lies in a fixed plane spanned by the tangent and normal vectors as seen in the images on the next page.


z
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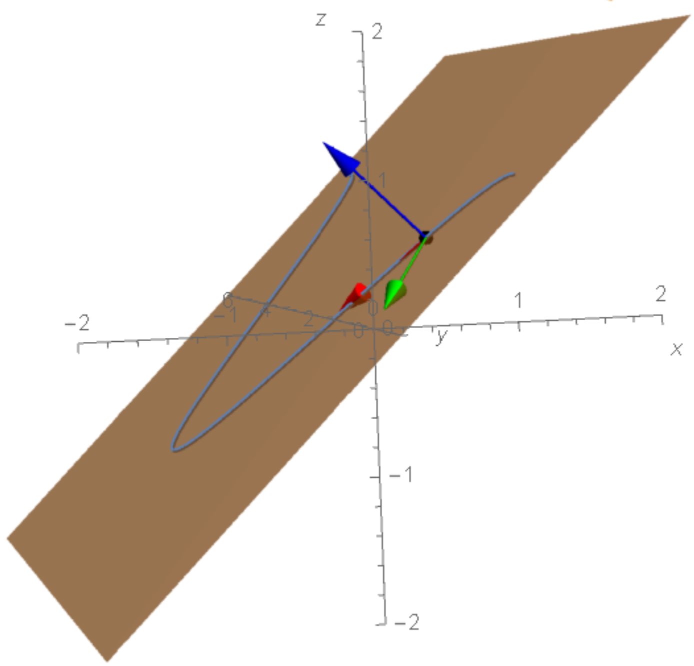
z
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-2

Problem 4. *Let a, b be positive real numbers. Let $\alpha(s)$, with $0 < s < a$ be a regular curve parametrized by arc length. Assume that $X : (0, a) \times (0, b) \longrightarrow \mathbb{R}^3$ given by $X(s, t) = \alpha(s) + t\dot{\alpha}(s)$ is a regular patch. (Dot is used to represent derivative with respect to s.) Prove that the curvature of α is different from zero for each $s \in (0, a)$.*

Proof. Seeing that $X_s(s, t) = \dot{\alpha}(s) + t\ddot{\alpha}(s)$ and $X_t(s, t) = \dot{\alpha}(s)$ we arrive at the equation

$$t\ddot{\alpha}(s) = X_s(s, t) - X_t(s, t)$$

Now since $X(s, t)$ is a regular surface patch we have X_s and X_t are linearly independent for all s, t in our domain. Thus $X_s(s, t) - X_t(s, t) \neq \bar{0}$ where $\bar{0}$ is the zero vector in $\mathbb{R}^3$. Taking the norm of both sides of our equation above we have

$$t|\ddot{\alpha}(s)| = |X_s(s, t) - X_t(s, t)| \neq 0$$

As this holds for all s, t in our domain we conclude the curvature of α is different from zero for each $s \in (0, a)$

$$\kappa(s) = |\ddot{\alpha}(s)| \neq 0$$

□

Problem 5. *Give a parametrization of a Möbius strip as the one shown below that is a portion of a ruled surface (it is exhausted by straight segments). Use Mathematica to produce the graph using your parametrization.*

Solution. A ruled surface is of the form $X(s,t) = \alpha(t) + s\beta(t)$ with $t \in I$ and $s \in (-\infty, \infty)$. Let

$$\alpha(t) = (\cos t, 0, \sin t)$$

with $t \in [0, 2\pi]$ be the unit circle in the xz-plane. Computing the T, N, B frame

$$\begin{aligned} T(t) &= \frac{\alpha'}{|\alpha'|} = (-\sin t, 0, \cos t) \\ B(t) &= \frac{\alpha' \times \alpha''}{|\alpha' \times \alpha''|} = (0, -1, 0) \\ N(t) &= B \times T = (-\cos t, 0, -\sin t) \end{aligned}$$

Using the plane spanned by N and B we define

$$\beta(t) = \cos(\frac{t}{2})N(t) + \sin(\frac{t}{2})B(t)$$

Note the $\frac{t}{2}$ is what gives us the half twist in the Möbius strip. Thus the Möbius strip can be parametrized by

$$\begin{aligned} X(s,t) &= \alpha(t) + s\beta(t) \\ &= (\cos t, 0, \sin t) + s\left[\cos(\frac{t}{2})N(t) + \sin(\frac{t}{2})B(t)\right] \end{aligned}$$

as seen in the image below for $t \in [0, 2\pi]$, $s \in [-.4, .4]$.

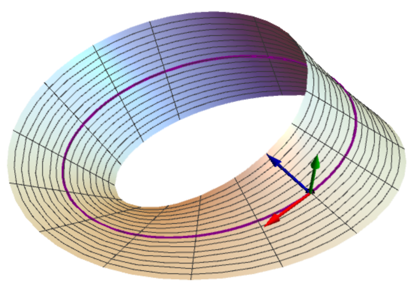


Figure 3: The Möbius strip showing the curve $\alpha(t)$ in purple and the T, N, B frame at a point.

Problem 6. *A surface* $\mathcal{M}$ *is the trace of the regular patch* $X : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ *given by*

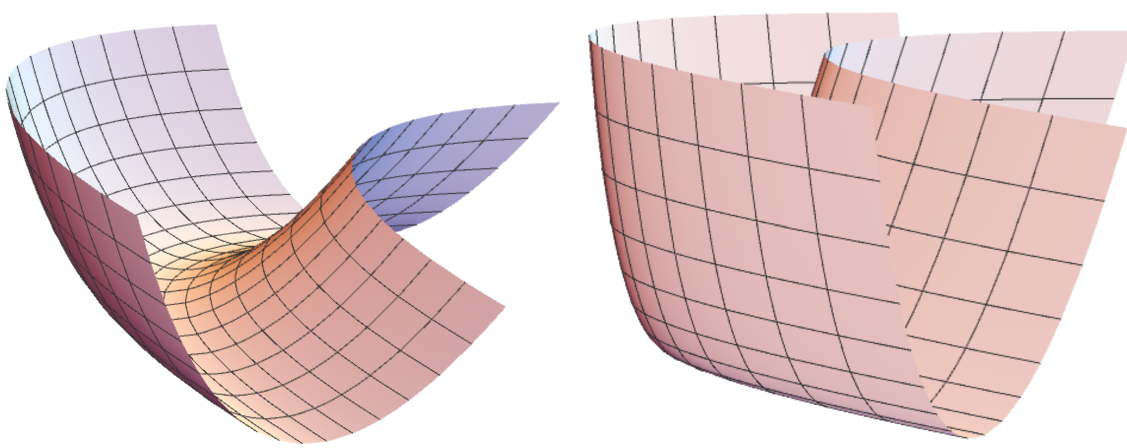
$$X(u,v) = (u^2 - 2v, u, u + v^2).$$

Compute the coefficients of the first fundamental form of $\mathcal{M}$*.*

Solution. The coefficients of the first fundamental form are given below after showing the necessary computations.

$$\begin{aligned}
X(u,v) &= (u^2 - 2v, u, u + v^2) \\
X_u(u,v) &= (2u, 1, 1) \\
X_v(u,v) &= (-2, 0, 2v) \\
E(u,v) &= X_u(u,v) \cdot X_u(u,v) = 4u^2 + 2 \\
F(u,v) &= X_u(u,v) \cdot X_v(u,v) = -4u + 2v \\
G(u,v) &= X_v(u,v) \cdot X_v(u,v) = 4 + 4v^2
\end{aligned}$$

The coefficients are E, F, G. See images of the surface below.



Problem 7. *Consider the surface $\mathcal{M}$ that is the trace of the patch $X : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ given by*

$$X(u,v) = \left(u^2,\, u-v,\, u+v\right).$$

Find the principal curvatures of the surface $\mathcal{M}$ at the point $(0,-1,1)$.

Solution. To compute the principal curvatures of the surface at $X(0,1) = (0,-1,1)$ we solve the quadratic equation $\det(II - \lambda I) = 0$ for λ where I and II are the matrices of the first and second fundamental form. We begin by carrying out the necessary computations.

$$\begin{aligned}
X &= \left(u^2,\, u-v,\, u+v\right) \\
X_u &= (2u, 1, 1) \\
X_v &= (0, -1, 1) \\
X_{uu} &= (2, 0, 0) \\
X_{uv} &= (0, 0, 0) \\
X_{vv} &= (0, 0, 0) \\
\eta &= \frac{X_u \times X_v}{|X_u \times X_v|} = \frac{1}{\sqrt{2u^2+1}}\,(1, -u, -u) \\
E &= X_u \cdot X_u = 4u^2 + 2 \\
F &= X_u \cdot X_v = 0 \\
G &= X_v \cdot X_v = 2 \\
L &= X_{uu} \cdot \eta = \frac{2}{\sqrt{2u^2+1}} \\
M &= X_{uv} \cdot \eta = 0 \\
N &= X_{vv} \cdot \eta = 0
\end{aligned}$$

Evaluating at the point $u = 0$, $v = 1$ we have

$$I = \begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$II = \begin{pmatrix} L & M \\ M & N \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$$

This gives us

$$0 = \det(II - \lambda I) = \begin{vmatrix} 2-2\lambda & 0 \\ 0 & -2\lambda \end{vmatrix} = 4\lambda(\lambda - 1)$$

Therefore the principal curvatures at the point $X(0,1) = (0,-1,1)$ are $\kappa_1 = 0$, $\kappa_2 = 1$.

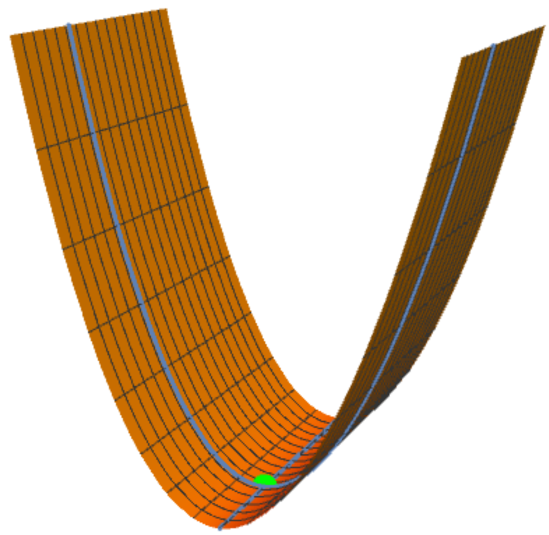
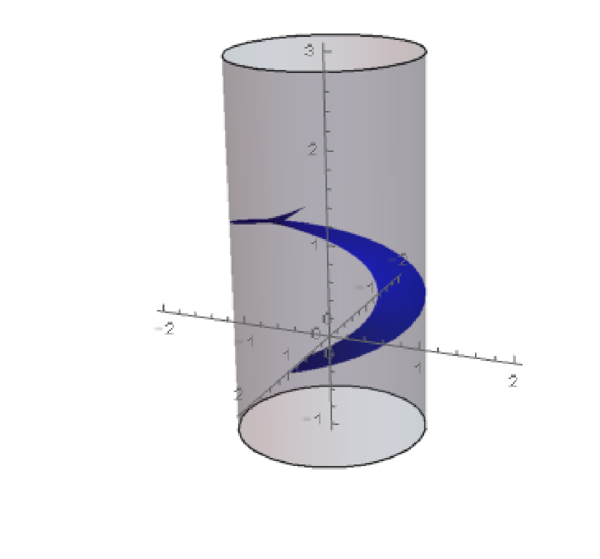
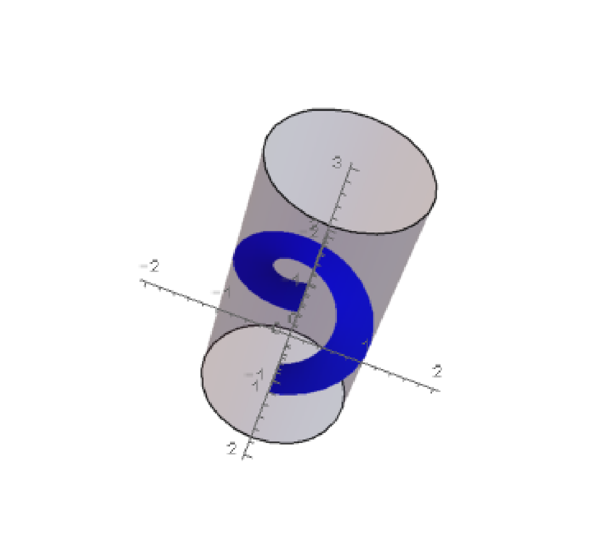


Figure 4: The surface from Problem 7 showing the lines of curvature at the point $(0, -1, 1)$

Problem 8. *The pictures below show a helicoidal ramp enclosed in a cylinder of radius one from different perspectives. The ramp can be parametrized by* $X(u,v) = (u\cos v, u\sin v, \frac{v}{4})$ *with* $0.25 < u < 1$, $0 < v < 2\pi$. *Find a double integral that represents the area of the ramp. You do not need to compute the integral but you must simplify the integrand (the expression inside the integral) as much as possible.*





Solution. We compute the coefficients of the first fundamental form $E = X_u \cdot X_u$, $F = X_u \cdot X_v$, and $G = X_v \cdot X_v$ where X_u, X_v are the partial derivatives with respect to u and v. The area of a regular patch is given by $\int_\Omega \sqrt{EG - F^2}\, du\, dv$.

$$\begin{aligned} X &= (u\cos v, u\sin v, \frac{v}{4}) & \qquad E &= 1 \\ X_u &= (\cos v, \sin v, 0) & F &= 0 \\ X_v &= (-u\sin v, u\cos v, \frac{1}{4}) & G &= u^2 + \frac{1}{16} \end{aligned}$$

The area is given by

$$\frac{1}{4}\int_0^{2\pi}\int_{\frac{1}{4}}^{1} \sqrt{16u^2+1}\, du\, dv \approx 3.19884$$