Convex Hulls and Partitions of Sets of Points

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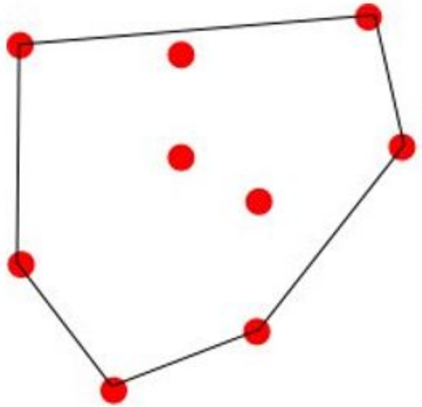
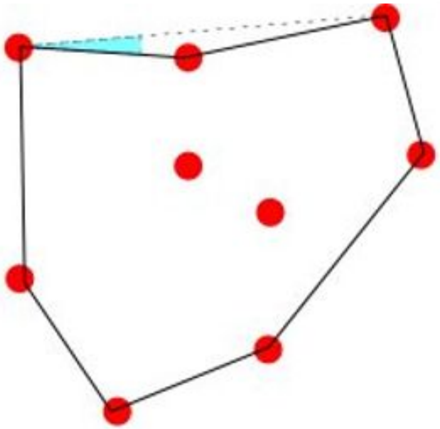
Convex Set

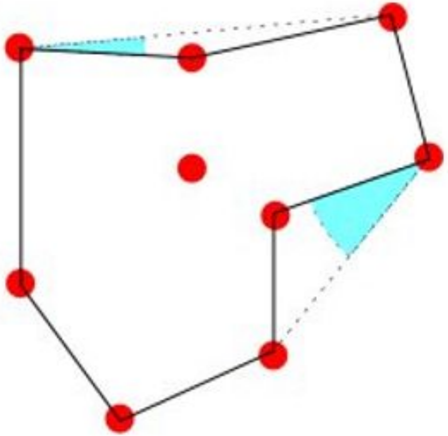
- A set of points is defined to be convex if it contains the line segments connecting each pair of its points.
- In other words, it is closed under convex combinations (linear combinations with non-negative coefficients that sum to 1).



Convex



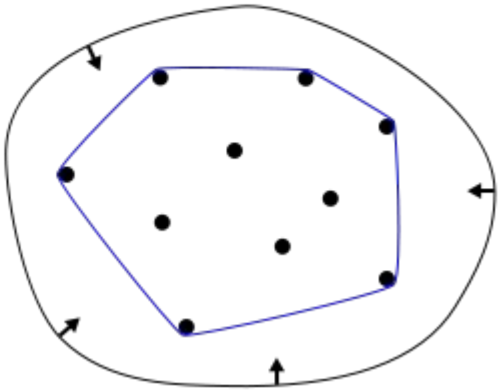
Not Convex



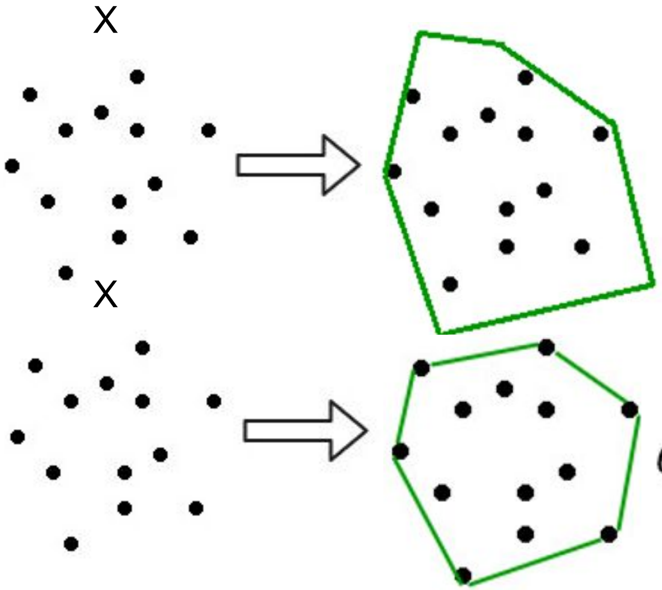
Not Convex

Convex Hull

- The closure of a set of points in Euclidean space.
- The set of all convex combinations, denoted $Conv(X)$.
- It is the *smallest* convex set containing all points.



Convex Hull



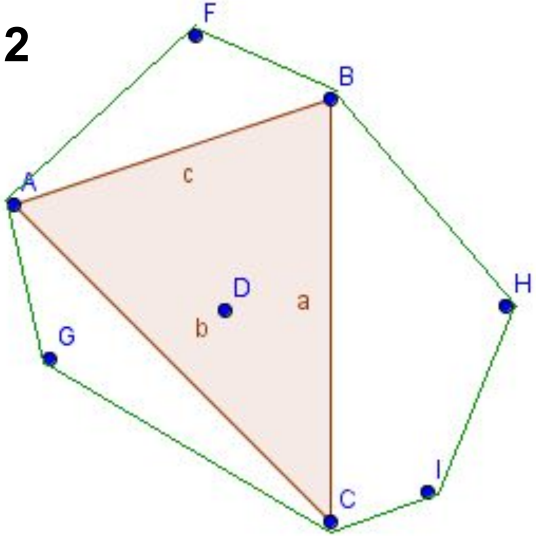
This is a convex set containing X, but it is *not the smallest.*

$Conv(X)$

1907 - Constantin Carathéodory's Theorem

- Carathéodory's theorem states that if a point *x* of $\mathbf{R}^d$ lies in the convex hull of a set *P*, then *x* can be written as the convex combination of at most *d* + 1 points in P.

$\mathbb{R}^2$



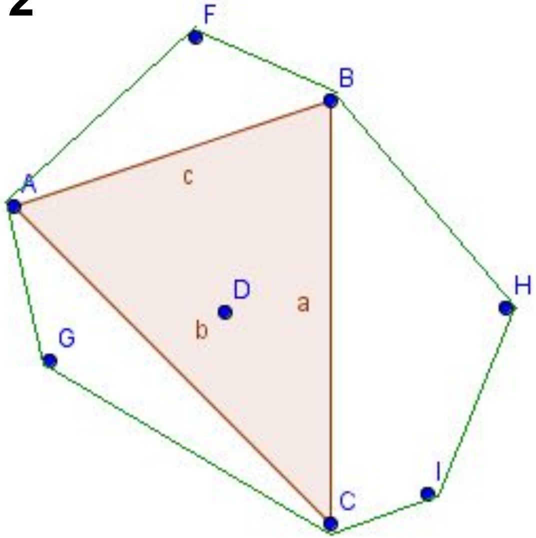
$$Point\ x = D$$

$$P = \{A, F, B, H, I, C, G\}$$

➔ So point D can be written as a convex combination of d + 1 = 3 points.

1907 - Constantin Carathéodory's Theorem (ex.)

$\mathbb{R}^2$



Writing D as a convex combination of $d + 1$ points

Points:

$A(1,4), B(4,5), C(4,1), D(2,3)$

Equation to solve:

$a(1,4) + b(4,5) + c(4,1) = (2,3)$

Solve matrix:

$$\begin{bmatrix} 1 & 4 & 4 & 2 \\ 4 & 5 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \frac{-16}{11} & \frac{2}{11} \\ 0 & 1 & \frac{15}{11} & \frac{5}{11} \end{bmatrix}$$

Solution:

$a = \frac{16}{11}c + \frac{2}{11},$

$b = \frac{-15}{11}c + \frac{5}{11},$

Constraint:

$a + b + c = 1$

Solving for a, b, and c:

$a = \frac{2}{3}, b = 0, c = \frac{1}{3}$

Checking Solution:

$\frac{2}{3}(1,4) + 0(4,5) + \frac{1}{3}(4,1) = (2,3)$

➔ It works!

1921 - Johann Radon’s Theorem

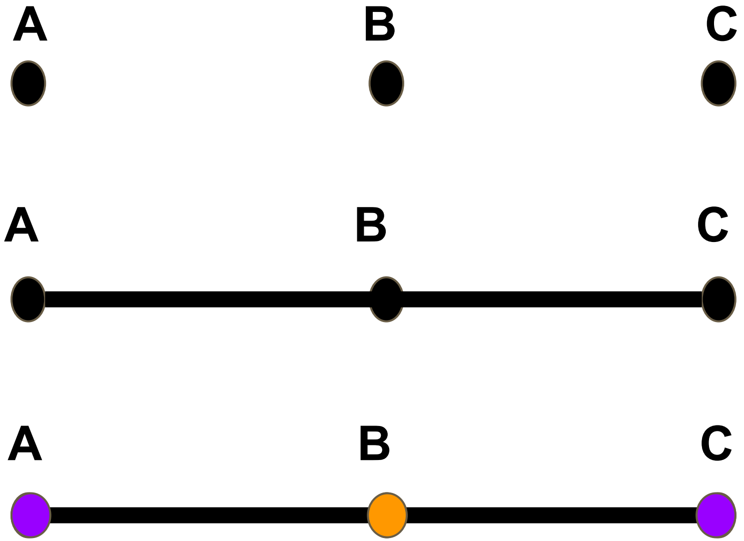
- Radon's theorem on convex sets, published by Johann Radon in 1921.
- Any set of d + 2 points in $\mathbf{R}^d$ can be partitioned into two disjoint sets whose convex hulls have a non-empty intersection.

    ➔ Let’s look at some specific examples in:

$\mathbb{R}$ $\mathbb{R}^2$ $\mathbb{R}^3$

1921 - Johann Radon’s Theorem in $\mathbb{R}$

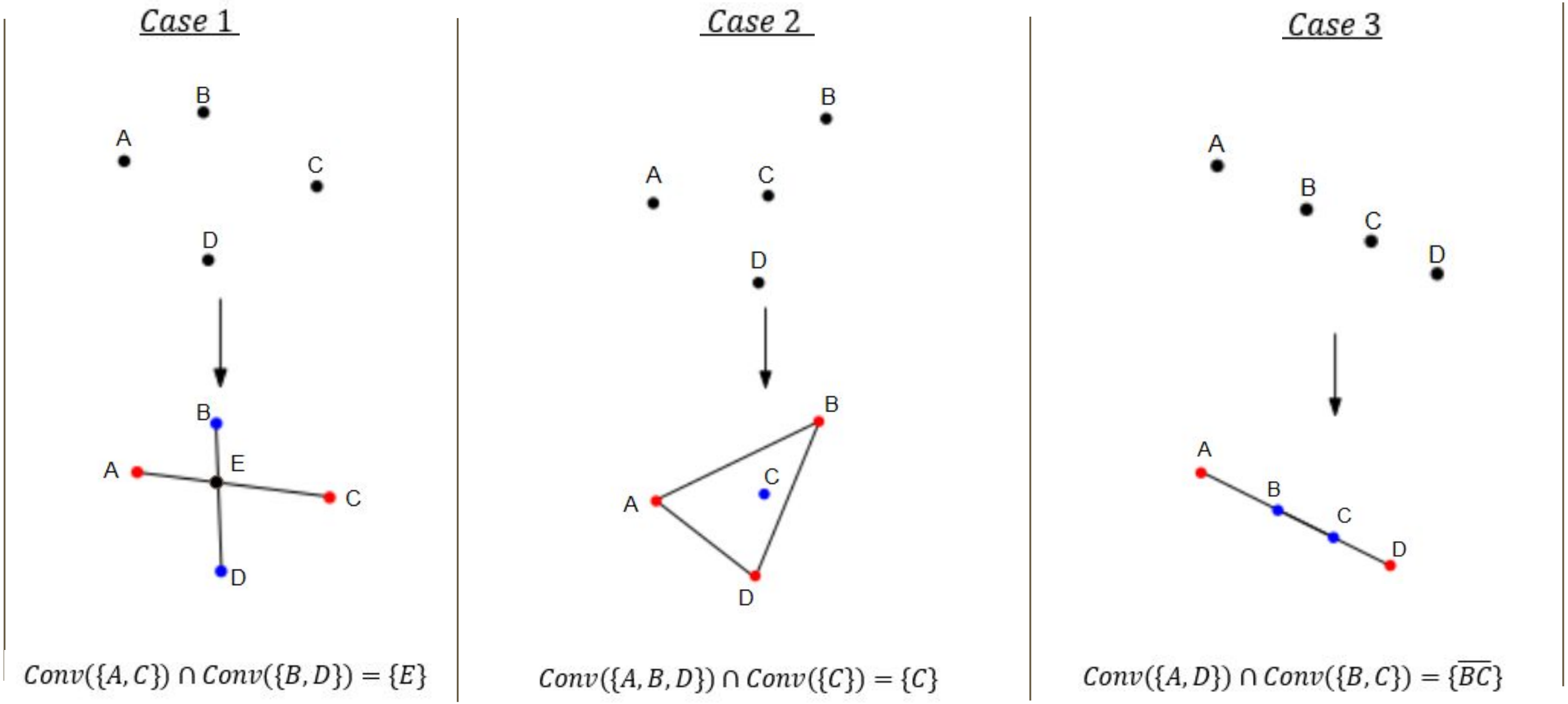
We need d + 2 = 3 points.



$$Conv(\{A, C\}) \cap Conv(\{B\}) = \{B\}$$

1921 - Johann Radon’s Theorem in $\mathbb{R}^2$

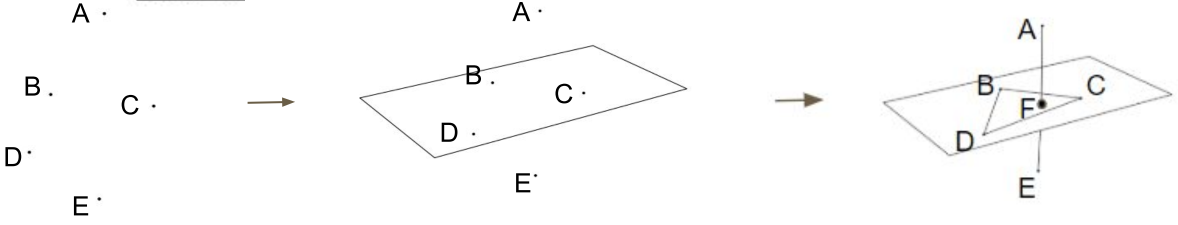
We need d + 2 = 4 points.



1921 - Johann Radon's Theorem in $\mathbb{R}^3$

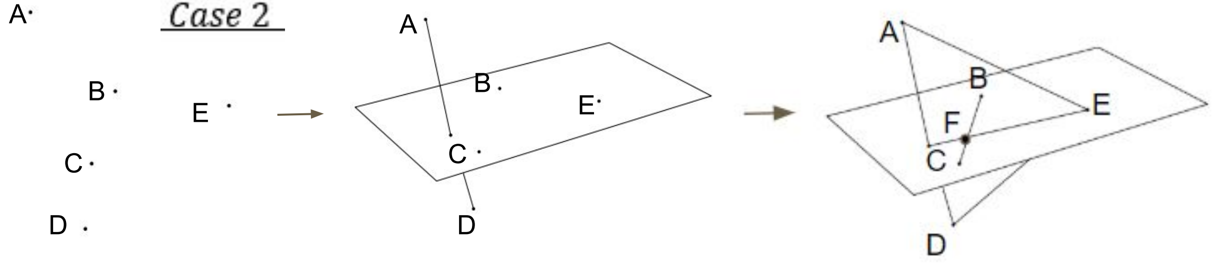
We need d + 2 = 5 points.

Case 1



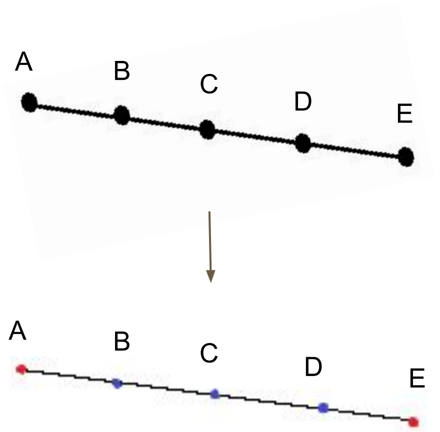
$$Conv(\{B,C,D\}) \cap Conv(\{A,E\}) = \{F\}$$

Case 2



$$Conv(\{A,D,E\}) \cap Conv(\{B,C\}) = \{F\}$$

Case 3



$$Conv(\{A,E\}) \cap Conv(\{B,C,D\}) = \{\overline{BD}\}$$

1966 - Helge Tverberg's Theorem

- Tverberg's Theorem is a generalization of Radon's Theorem.
- Given at least $(d + 1)(r - 1) + 1$ points in $\mathbf{R}^d$, we can always partition these points into r sets, such that the convex hulls of these sets have a non-empty intersection.

➔ Let's look at some specific examples in:

$$\mathbb{R} \qquad \mathbb{R}^2$$

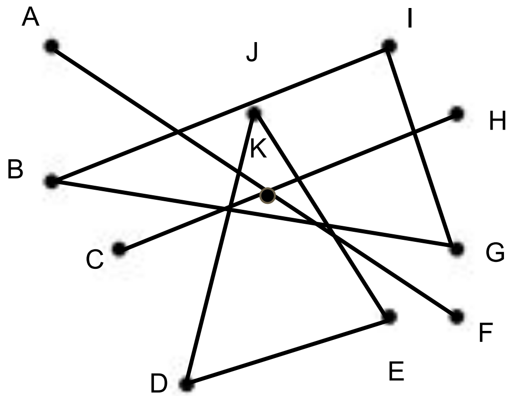
1966 - Helge Tverberg's Theorem

Example 1

$When\ d = 2\ and\ r = 4$

$(d+1)(r-1)+1 = 10$

$So\ we\ need\ 10\ points$



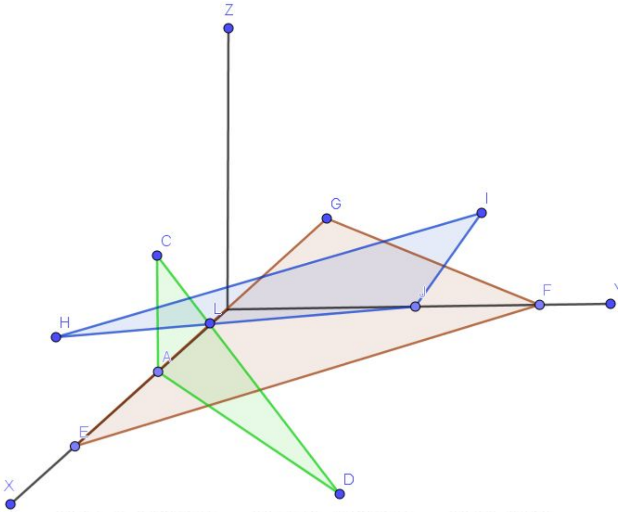
$Conv(\{B,I,G\}) \cap Conv(\{D,J,E\}) \cap Conv(\{C,H\}) \cap Conv(\{A,F\}) = \{K\}$

Example 2

$When\ d = 3\ and\ r = 3$

$(d+1)(r-1)+1 = 9$

$So\ we\ need\ 9\ points$



$Conv(\{C,A,D\}) \cap Conv(\{E,G,F\}) \cap Conv(\{H,I,J\}) = \{L\}$

Questions?

Works Cited

- https://i.stack.imgur.com/xSA15.jpg
- https://en.wikipedia.org/wiki/Convex_hull
- https://en.wikipedia.org/wiki/Helly%27s_theorem#Proof
- https://en.wikipedia.org/wiki/Tverberg%27s_theorem
- https://en.wikipedia.org/wiki/Radon%27s_theorem
- https://en.wikipedia.org/wiki/Carath%C3%A9odory%27s_theorem_(convex_hull)
- https://perso.esiee.fr/~mustafan/TechnicalWritings/math-lec2.pdf